$\vec{F} = \hat{i}x + \hat{j}y$

3-D particles

Conservative

$\boxed{3.1}$

(a) $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial y}{\partial z} - 0\right) - \hat{j}\left(\frac{\partial x}{\partial z} - 0\right) + \hat{k}\left(\frac{\partial x}{\partial y} - \frac{\partial y}{\partial x}\right)$$

$$= 0 + 0 + \hat{k}(0 - 0) = \underline{\underline{\vec{0}}}$$

10

(b) $\vec{F} = \hat{i}y + \hat{j}x$

Conservative

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y & x & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial x}{\partial z} - 0\right) - \hat{j}\left(\frac{\partial y}{\partial z} + 0\right) + \hat{k}\left(\frac{\partial y}{\partial y} - \frac{\partial x}{\partial x}\right) = \underline{\underline{\vec{0}}}$$

(c) $\vec{F} = (y, -x, 0)$

not

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y & -x & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{vmatrix} = 0 + 0 + \hat{k}\left(\frac{\partial y}{\partial y} + \frac{\partial x}{\partial x}\right) = \underline{2\hat{k}}$$

(d) $\vec{F} = (xy, yz, zx)$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xy & yz & zx \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{vmatrix} = \hat{i}\left(\frac{\partial yz}{\partial x} - \frac{\partial zx}{\partial y}\right) - \hat{j}\left(\frac{\partial xy}{\partial x} - \frac{\partial zx}{\partial x}\right)$$

$$+ \hat{k}\left(\frac{\partial xy}{\partial y} - \frac{\partial yz}{\partial x}\right)$$

$$= \hat{i}(0) + -\hat{j}(y - \cancel{z}) + \hat{k}(x - 0) \neq \vec{0}$$ not conservative

$\boxed{3.2}$ Find c so $\vec{F}$ is conservative

(a)

5/5

$$\vec{F} = (xy, cx^2, 0)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xy & cx^2 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial cx^2}{\partial z} + 0\right) + \hat{j}\left(\frac{\partial xy}{\partial z} - 0\right) + \hat{k}\left(\frac{\partial xy}{\partial \cancel{y}} - \frac{\partial cx^2}{\partial x}\right)$$

$$= \hat{i}\,0 - \hat{j}\,0 + \hat{k}\left(\cancel{x} - c\,2x\right) \quad \boxed{c = \frac{1}{2}}$$

(b)

$$\vec{F} = \left(\frac{z}{y}, \frac{cxz}{y^2}, \frac{x}{y}\right)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ z/y & cxz/y^2 & x/y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial\, cxz/y^2}{\partial z} - \frac{\partial\, x/y}{\partial y}\right) \hat{j}\left(\frac{\partial\, z/y}{\partial z} - \frac{\partial\, x/y}{\partial x}\right)$$

$$+ \hat{k}\left(\frac{\partial\, z/y}{\partial y} - \frac{\partial\, cxz/y^2}{\partial x}\right)$$

$$= \left(\frac{cx}{y^2} + \frac{x}{y^2}\right)\hat{i} - \hat{j}\left(\frac{1}{y} - \frac{1}{y}\right) + \hat{k}\left(\frac{-z}{y^2} - \frac{cz}{y^2}\right) \quad c = -1$$

3.3 Find $\vec{F}$ for given $V(x,y,z)$

(a) 10

$V = xyz$

$-\vec{\nabla} V = \vec{F} = \hat{i}\frac{\partial xyz}{\partial x} + \hat{j}\frac{\partial xyz}{\partial y} + \hat{k}\frac{\partial xyz}{\partial z}$

$$= (\hat{i}, \hat{j}, \hat{k})\begin{pmatrix} -yz \\ -xz \\ -xy \end{pmatrix}$$

(b) $V = K(x^\alpha + y^\beta + z^\gamma)$

$\vec{F} = -\vec{\nabla} V = -(K\alpha x^{\alpha-1})\hat{i} + -(K\beta y^{\beta-1})\hat{j} + -(K\gamma z^{\gamma-1})\hat{k}$

$$\vec{F} = (\hat{i}, \hat{j}, \hat{k})\begin{pmatrix} -K\alpha x^{\alpha-1} \\ -K\beta y^{\beta-1} \\ -K\gamma z^{\gamma-1} \end{pmatrix}$$

(c) $V = Kx^\alpha y^\beta z^\gamma$

$-\vec{\nabla} V = -\hat{i}(Ky^\beta z^\gamma \alpha x^{\alpha-1}) - \hat{j}(Kx^\alpha \beta y^{\beta-1} z^\gamma) - \hat{k}(Kx^\alpha y^\beta \gamma z^{(\gamma-1)})$

$$F = (\hat{i}, \hat{j}, \hat{k})\begin{pmatrix} -Ky^\beta z^\gamma \alpha x^{\alpha-1} \\ -Kx^\alpha \beta y^{\beta-1} z^\gamma \\ -Kx^\alpha y^\beta \gamma z^{\gamma-1} \end{pmatrix} = -V(\hat{i}, \hat{j}, \hat{k})\begin{pmatrix} \alpha/x \\ \beta/y \\ \gamma/z \end{pmatrix}$$

(d) $V = Ke^{(\alpha x + \beta y + \gamma z)}$

$\vec{F} = -\vec{\nabla} V = \hat{i}(-V\alpha) + \hat{j}(-V\beta) + \hat{k}(-V\gamma)$

$$= -V(\hat{i}, \hat{j}, \hat{k})\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$\boxed{3.4}$ Find $V(x,y,z)$

Find the potential function for cons. $\vec{F}$ in 3.1

~~6~~ ~~10~~

3|5

(a) $\vec{F} = (\hat{\imath}, \hat{\jmath}, \hat{k})\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$\int \vec{F} \cdot d\vec{r} = \int dV$

$\int (x, y, 0)(dx, dy, dz) = \int dV$

$= \int x\,dx + \int y\,dy = V + c$

$\boxed{\frac{x^2}{2} + \frac{y^2}{2} - c = V}$ ✓

$V = -\frac{1}{2}(x^2+y^2)$

(b) $\vec{F} = (\hat{\imath}, \hat{\jmath}, \hat{k})\begin{pmatrix} y \\ x \\ 0 \end{pmatrix}$

$\int \vec{F} \cdot d\vec{r} = V + c$

$\int y\,dx + x\,dy + 0 = V + c$

$-\cancel{3}$

$xy + \cancel{xy}$ $\boxed{2xy + c = V}$ ✗

no

$V = -xy$

$\int \vec{F} \cdot d\vec{r}$ this integral is line integral remember?

V is best found by inspection

(c) $\vec{F} = (yz, xz, yx)$

$\int (yz\,dx + \overset{no}{\cancel{xz\,dy}} + \cancel{yx\,dz}) = V + c$

$yzx + \cancel{xzy} + \cancel{yxz}$ $\boxed{3xyz + c = V}$ ✗

-1

$V = -xyz$

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PHYSICS 321 TAKE-HOME QUIZ Sept. 18, 1987

Due: 1:10 pm, Sept. 21

Open book - you may use your text, notes, and problem solutions. You may also use math tables. You may not use any other books or consult with anyone except Dr. Evenson.

(10) 1. The force acting on a particle of mass m is given by

$$F = kv^2x$$

in which k is a constant. The particle passes through the origin with speed v_0 at time $t=0$. Find x as a function of t.

$$\vec{F} = Kv^2x = m\ddot{x} = mv\frac{dv}{dx}$$

so

$$Kv^2x = mv\frac{dv}{dx}$$

$$(dx)Kx = \frac{mv\,dv}{v^2}$$

$$\int_0^x x\,dx = \frac{m}{K}\int_{v_0}^{v}\frac{dv}{v}$$

$$\frac{x^2}{2} = \frac{m}{K}\left(\ln|v| - \ln|v_0|\right)$$

$$\frac{x^2}{2} = \frac{m}{K}\ln\left|\frac{v}{v_0}\right|$$

$$\frac{Kx^2}{2m} = \ln\left|\frac{v}{v_0}\right|$$

$$\boxed{v_0 e^{\frac{Kx^2}{2m}} = v}$$

$$v = \dot{x} = v_0 e^{\frac{Kx^2}{2m}}$$

$$\int_0^x e^{-\frac{Kx^2}{2m}}dx = \int_0^t v_0\,dt$$

$$v_0 t = \int_0^x e^{-\frac{Kx^2}{2m}}dx$$

$$t = \frac{1}{v_0}\int_0^x e^{-\frac{Kx^2}{2m}}dx$$

$$e^u = \sum \frac{u^n}{n!} \qquad u = -\frac{K}{2m}x^2$$

$$\int e^u dx = \int \sum \frac{(-1)^n\left(\frac{K}{2m}\right)^n x^{2n}}{n!}dx$$

$$\boxed{t(x) = \sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{K}{2m}\right)^n x^{2n+1}}{n!\,(2n+1)}}$$

So the best I can do is t(x).

10/10

Assign #4 Chpt 3. 5, 7, 8, 11, 12, 15, 16, 20

$\boxed{\text{3-D motion Continued}}$

$\boxed{3.5}$ a particle of mass m moves in $\vec{F}$ given by $V = xyz$

5/5 @ (0,0,0) $v = v_0$ find $\underline{v}$ @ (2,2,2)

Total energy

$$E_T = KE + PE = \frac{1}{2}mv^2 + xyz$$

@ (0,0,0) $E_T = \frac{1}{2}mv_0^2 + (0 \cdot 0 \cdot 0) = \frac{1}{2}mv_0^2$

$$E_T = \frac{1}{2}mv^2 + xyz = \frac{1}{2}mv_0^2$$

@(2,2,2) $v^2 = \left(\frac{1}{2}mv_0^2 - 2 \cdot 2 \cdot 2\right)\frac{2}{m}$

$$\boxed{v = \sqrt{v_0^2 - \frac{16}{m}}}$$

$\boxed{3.7}$ Show variation with gravity for height "z" accounted for by using potential

7

$$V = mgz\left(1 - \frac{z}{R}\right)$$

$V = -\frac{GMm}{r}$ let $r = R + z$ $V = -\frac{GMm}{R+z} = \frac{-GMm}{\left(1+\frac{z}{R}\right)R}$

mult top & bottom by R $V = -\underbrace{\left(\frac{GM}{R^2}\right)}_{g} m \boxed{R\left(1+\frac{z}{R}\right)^{-1}}$

expand : $\frac{1}{1+x} \approx 1 - x + x^2$ so $R\left(1+\frac{z}{R}\right)^{-1} \approx R\left(1 - \frac{z}{R} + \frac{z^2}{R^2}\right)$

$V = -gmR + gmz - gm\frac{z^2}{R}$ by redefining zero point to be surface of earth.

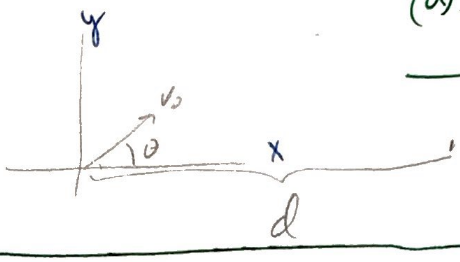
$$\boxed{V = gmz\left(1 - \frac{z}{R}\right)}$$

unfinished

-3

$\boxed{3.8}$ Projectile problems

(a) show $d = \dfrac{v_0^2 \sin 2\theta}{g}$



10

y: $v_0 \sin\theta = v_{vert.}$ time to peak

$V_f = at_f + v_0$

$0 = -gt + v_{0v}$

$t = \dfrac{v_{0v}}{g}$ $t = \dfrac{v_0 \sin\theta}{g}$

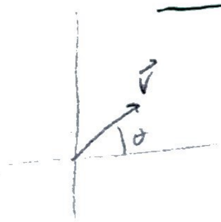
x: $v_0 \cos\theta = v_{0h}$

$\underline{d = vt}$

up and back down

$d = (v_0 \cos\theta)\left(2\dfrac{v_0 \sin\theta}{g}\right) = \dfrac{2}{g} v_0^2 \cos\theta \sin\theta = \underline{\underline{\dfrac{v_0^2 \sin 2\theta}{g}}}$ ✓

(b) show decrease in horizontal range $\approx \dfrac{4 v_0^3 \gamma \sin\theta \sin 2\theta}{3g^2}$



$\vec{v}_0 = (v_0 \cos\theta, v_0 \sin\theta)$

eq. of motion $m\ddot{\vec{r}} = -m\gamma\vec{v} - mg\hat{j}$

$m\ddot{x} = -m\gamma v_x$

$m\ddot{y} = -m\gamma v_y - mg$

or

$\ddot{x} = -\gamma v_x$

$\ddot{y} = -\gamma v_y - g$

integrate

$$\int_{v_{x_0}}^{v_x} \frac{dv_x}{v_x} = -\gamma \int_0^t dt \qquad \ln|v_x| \Big]_{v_{x_0}}^{v_x} = -\gamma t \quad \rightarrow \quad \ln\left|\frac{v_x}{v_{x_0}}\right| = -\gamma t \qquad \underline{v_x = v_{x_0} e^{-\gamma t}}$$

$u = -\gamma v_y - g$

$du = -\gamma dv_y$

$$\int_{v_{y_0}}^{v_y} \frac{dv_y}{-\gamma v_y - g} = \int_0^t dt \quad \rightarrow \quad \frac{1}{-\gamma} \ln|-\gamma v_y - g| \Big]_{v_{y_0}}^{v_y} = t \quad \rightarrow \quad \ln\left|\frac{-\gamma v_y - g}{-\gamma v_{y_0} - g}\right| = -\gamma t$$

$+\gamma v_y + g$

$= (\gamma v_{y_0} + g) e^{-\gamma t}$

$$\underline{v_y = \left(v_{y_0} + \frac{g}{\gamma}\right) e^{-\gamma t} - \frac{g}{\gamma}}$$

(3.8 cont.)

$$\text{so} \quad v_y = \underbrace{\left(v_{y_0} + \frac{g}{\gamma}\right)}_{K} e^{-\gamma t} - \frac{g}{\gamma} = \dot{y}$$

$$\text{so} \int_0^y dy = \int_0^t \left(K e^{-\gamma t} - \frac{g}{\gamma}\right) dt = \frac{K e^{-\gamma t}}{-\gamma} - \frac{g}{\gamma} t \Big]_0^t = \frac{K e^{-\gamma t}}{-\gamma} - \frac{g t}{\gamma} - \frac{K}{-\gamma}$$

therefore

$$\boxed{y(t) = \frac{K}{\gamma}\left(1 - e^{-\gamma t}\right) - \frac{g}{\gamma} t}$$

$$v_x = v_{x_0} e^{-\gamma t} = \dot{x}$$

$$\int_0^x dx = v_{x_0} \int_{t=0}^t e^{-\gamma t} dt \rightarrow x = \frac{v_{x_0}}{-\gamma}\left(e^{-\gamma t} - 1\right) \quad \text{or}$$

$$\boxed{x = \frac{v_{x_0}}{\gamma}\left(1 - e^{-\gamma t}\right)}$$

When $y(t) = 0$ $x(t) = \text{range}$. $v_{x_0} = \dot{x}_0$, $v_{y_0} = \dot{y}_0$

from x when solve for t $\boxed{\frac{\ln}{-\gamma}\left(-\frac{x\gamma}{\dot{x}_0} + 1\right) = t}$

therefore

$$y = \frac{K}{\gamma}\left(1 - e^{-\gamma \frac{\ln}{-\gamma}\left(-\frac{x\gamma}{\dot{x}_0} + 1\right)}\right) - \frac{g}{\gamma}\left(\frac{\ln}{-\gamma}\left(-\frac{x\gamma}{\dot{x}_0} + 1\right)\right)$$

$$y = \frac{K}{\gamma}\left(1 - \left(1 - \frac{x\gamma}{\dot{x}_0}\right)\right) + \frac{g}{\gamma^2} \ln\left(1 - \frac{x\gamma}{\dot{x}_0}\right) \Longleftarrow$$

using $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$

$$y = +\frac{K}{\gamma}\frac{x\gamma}{\dot{x}_0} + \frac{g}{\gamma^2}\left[-\frac{x\gamma}{\dot{x}_0} - \frac{1}{2}\left(\frac{x\gamma}{\dot{x}_0}\right)^2 - \frac{1}{3}\left(\frac{x\gamma}{\dot{x}_0}\right)^3 - \dots\right\}$$

$$y \approx \frac{Kx}{\dot{x}_0} - \frac{gx}{\gamma\dot{x}_0} - \frac{gx^2}{2\dot{x}_0^2} - \frac{gx^3\gamma}{3\dot{x}_0^3} -$$

$y = 0$ at range.

$$\frac{Kx}{\dot{x}_0} - \frac{gx}{\gamma\dot{x}_0} - \frac{gx^2}{2\dot{x}_0^2} - \frac{gx^3\gamma}{3\dot{x}_0^3} = 0$$

$$\frac{x}{\dot{x}_0}\underbrace{\left(K - \frac{g}{\gamma} - \frac{g}{2}\left(\frac{x}{\dot{x}_0}\right) - \frac{g\gamma}{3}\left(\frac{x}{\dot{x}_0}\right)^2\right)}_{0} = 0 \quad \Rightarrow \quad \boxed{x = 0}$$

$$x^2\left(\frac{g\gamma}{3\dot{x}_0^2}\right) + x\left(\frac{g}{2\dot{x}_0}\right) + \left(\frac{g}{\gamma} - K\right) = 0 \qquad \frac{g}{\gamma} - \left(y_0 + \frac{g}{\gamma}\right) = -y_0$$

$$\boxed{x^2\left(\frac{g\gamma}{3\dot{x}_0^2}\right) + x\left(\frac{g}{2\dot{x}_0}\right) - y_0 = 0} \Rightarrow x^2 + x\left(\frac{3\dot{x}_0}{2\gamma}\right) - \frac{3y_0\dot{x}_0^2}{g\gamma} = 0$$

$$x = \frac{-\frac{3\dot{x}_0}{2\gamma} \pm \left(\frac{9\dot{x}_0^2}{4\gamma^2} - 4\left(1\left(-\frac{3y_0\dot{x}_0^2}{g\gamma}\right)\right)\right)^{\frac{1}{2}}}{2} = -\frac{3\dot{x}_0}{4\gamma} \pm \frac{3\dot{x}_0}{4\gamma}\left(1 + \underbrace{\left(\frac{4\gamma^2}{9\dot{x}_0^2}\right)\left(\frac{12y_0\dot{x}_0^2}{g\gamma}\right)}_{\frac{16\gamma y_0}{3g}}\right)^{\frac{1}{2}}$$

using the approximation $(1+x)^{\frac{1}{2}} \approx 1 + \frac{x}{2} - \frac{1}{8}x^2 + \dots$

for $x > 0$ we take $+$ of $\pm$

$$x \approx -\frac{3\dot{x}_0}{4\gamma} \pm \frac{3\dot{x}_0}{4\gamma}\left[1 + \frac{8\gamma y_0}{3g} - \left(\frac{16\gamma y_0}{3g}\right)^2\frac{1}{8}\right]$$

$$x' = \frac{2\dot{x}_0 y_0}{g} - \frac{3\cdot16\cdot16\,\gamma^2 y_0^2\dot{x}_0}{4\cdot8\cdot3\cdot3\,\gamma g^2\cdot8}$$

3.8 cont.

$$x' = \frac{2\dot{x}_0\dot{y}_0}{g} + \frac{4\cdot 2\gamma^2 \dot{y}_0^2\dot{x}_0}{3\gamma g^2} = \frac{2\dot{x}_0\dot{y}_0}{g} + \frac{8\gamma \dot{y}_0^2\dot{x}_0}{3g^2}$$

$$x' = \underbrace{\frac{2v_0\sin\theta\, v_0\cos\theta}{g}}_{\text{no wind resistance term}} - \frac{8\gamma v_0^3\sin^2\theta\cos\theta}{3g^2}$$

difference

$$\Delta x = \frac{8\gamma v_0^3(\sin\theta\cos\theta)\sin\theta}{3g^2} = \boxed{\frac{4\gamma v_0^3\sin 2\theta\sin\theta}{3g^2}}$$

$\boxed{3.11}$ write down eq. of motion for air resistance $\propto v^2$

Are they separated? show $\dot{x} = \dot{x}_0 e^{-\gamma s}$

① eq of motion

$$\boxed{m\ddot{\vec{r}} = -m\gamma \|\vec{v}\|^2 \left(\frac{\vec{v}}{\|\vec{v}\|}\right) - mg\hat{k}}$$

or $m\ddot{\vec{r}} = -m\gamma v\vec{v} - mg\hat{k}$

$$m(\hat{i},\hat{j},\hat{k})\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = -m\gamma v(\hat{i},\hat{j},\hat{k})\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} - mg(\hat{i},\hat{j},\hat{k})\begin{pmatrix}0\\0\\1\end{pmatrix}$$

components:

$$\left\{\begin{array}{l} m\ddot{x} = -m\gamma v\dot{x} \\ m\ddot{y} = -m\gamma v\dot{y} \\ m\ddot{z} = -m\gamma v\dot{z} - mg \end{array}\right\} \qquad v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{s}^2$$

not separated so far as $\dot{x}, \dot{y}, \dot{z}, x, y, z$ concerned, but with $\dot{s}$ (or v) they can be separated. see following

---

$$\ddot{x} = -\gamma v\dot{x} = -\gamma\dot{s}\dot{x}$$

$$\frac{d\dot{x}}{dt} = -\gamma\frac{ds}{dt}\dot{x} \qquad \int\frac{d\dot{x}}{\dot{x}} = -\int\gamma\, ds \Rightarrow \ln|\dot{x}|\Big]_{\dot{x}_0}^{\dot{x}} = -\gamma s\Big]_0^s$$

$$\ln\left|\frac{\dot{x}}{\dot{x}_0}\right| = -\gamma s \quad \text{or} \quad \boxed{\dot{x} = \dot{x}_0 e^{-\gamma s}}$$

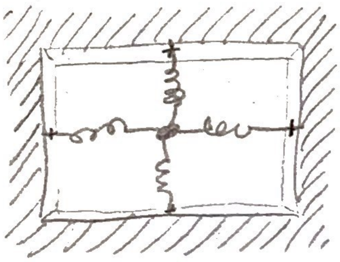
$\boxed{3.12}$ initial cond for 2d isotropic osc.

10

$x(0) = A$ $\qquad y(0) = B$ $\qquad$ Show: motion confined by $2A \text{ \& } 2(B^2+C^2)^{\frac{1}{2}}$

$\dot{x}(0) = 0$ $\qquad \dot{y}(0) = \omega C$

Find: $\psi(A,B,C)$



The eq. of motion for small displacements

$$m\ddot{x} = -kx \quad \Rightarrow \quad \ddot{x} + \omega^2 x = 0 \qquad \omega^2 = \frac{k}{m}$$

$$m\ddot{y} = -ky \quad \Rightarrow \quad \ddot{y} + \omega^2 y = 0$$

Soln's: $x(t) = a\cos\omega t + b\sin\omega t$ $\qquad \dot{x}(t) = -a\omega\sin\omega t + b\omega\cos\omega t$

$x(0) = \boxed{a \equiv A}$ $\qquad \dot{x}(0) = b\omega \equiv 0 \Rightarrow \boxed{b = 0}$

$y(t) = a'\cos\omega t + b'\sin\omega t$ $\qquad \dot{y}(t) = -\omega a'\sin\omega t + b'\omega\cos\omega t$

$y(0) = \boxed{a' \equiv B}$ $\qquad \dot{y}(0) = b'\omega \equiv \omega C \Rightarrow \boxed{b' = C}$

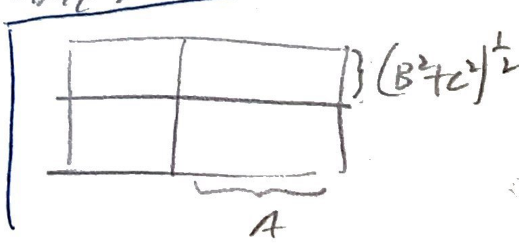
so

$$\boxed{\begin{aligned} x(t) &= A\cos\omega t \\ y(t) &= B\cos\omega t + C\sin\omega t \end{aligned}}$$

amplitude above or below equilibrium axis:

$\|x\| = \sqrt{A^2 + 0^2} = A$ $\qquad$ The rectangle is

$\|y\| = \sqrt{B^2 + C^2}$



(3.12 cont.)

Now Find $\psi(A,B,C)$.

from $x = A\cos\omega t$ $\quad t = \dfrac{\cos^{-1}\left(\frac{x}{A}\right)}{\omega}$

So $\quad y = B\cos\omega t + C\sin\omega t = B\cos\left(\dfrac{\cos^{-1}\left(\frac{x}{A}\right)}{\omega}\right) + C\sqrt{1-\cos^2\left(\dfrac{\cos^{-1}\left(\frac{x}{A}\right)}{\omega}\right)}$

$$= \frac{Bx}{A} + C\sqrt{1-\left(\frac{x}{A}\right)^2} \Rightarrow \left(y - \frac{Bx}{A}\right)^2 = C^2\left(1-\frac{x^2}{A^2}\right)$$

or $\quad y^2 - \dfrac{2Byx}{A} + \dfrac{B^2x^2}{A^2} + \dfrac{C^2}{A^2}x^2 - C^2 = 0$

$$\boxed{y^2 + x^2\left(\frac{B^2+C^2}{A^2}\right) - \frac{2Byx}{A} - C^2 = 0}$$ equation of path.

For a quadratic equation of form

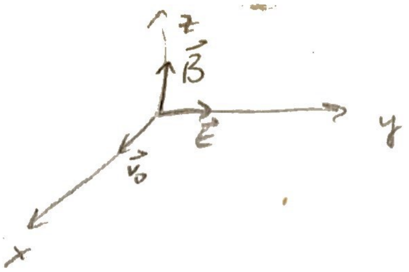
$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

the angle of rotation is $\quad \cot 2\psi = \dfrac{a-c}{2b}$

So

$$\psi = \frac{1}{2}\cot^{-1}\frac{\left(\frac{B^2+C^2}{A^2}\right) - 1}{2\left(\frac{-B}{A}\right)} = \boxed{\frac{1}{2}\cot^{-1}\frac{A^2-B^2-C^2}{2BA} = \psi}$$

$\boxed{3.15}$



@ $t = 0$, $(0,0,0)$ $\vec v = v_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

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Find resulting motion of particle ; show its cycloidal.

let $\vec E = \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix}$ & $\vec B = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$ $\quad \vec F = q(\vec v \times \vec B + \vec E)$

$$m \begin{pmatrix} \ddot x \\ \ddot y \\ \ddot z \end{pmatrix} = q\left[\begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \dot x & \dot y & \dot z \\ 0 & 0 & B \end{vmatrix} + \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix} \right] = q\left[B(\dot y \hat\imath - \hat\jmath \dot x) + \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix} \right]$$

$$= q\left[B \begin{pmatrix} \dot y \\ -\dot x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix} \right]$$

separating into eq. components.

$$\begin{cases} m\ddot x = qB\dot y \\ m\ddot y = -q(B\dot x - E) \\ m\ddot z = 0 \end{cases}$$

- $m\ddot z = 0$ $\quad m\int d\dot z = 0\int dt \rightarrow \dot z - \dot z_0 = 0 \quad \dot z_0 = 0$ so

$\dot z = 0 \Rightarrow z - z_0 = 0$

but $z_0 = 0$

so $\boxed{z = 0}$

$\Rightarrow$

(cont.)

(3.15 cont.)

- $m\ddot{x} = qB\dot{y}$

$$m\int_{\dot{x}_0}^{\dot{x}} d\dot{x} = qB\int_{y_0}^{y} dy \rightarrow m(\dot{x}-\dot{x}_0) = qB(y-y_0)$$

$\dot{x}_0 = v_0$

$y_0 = 0$

So $m(\dot{x}-v_0) = qBy$ or $\boxed{\dot{x} = \frac{qB}{m}y + v_0}$

- $m\ddot{y} = q(E - B\dot{x})$

$$m\ddot{y} - qE + qB\left(\frac{qBy}{m} + v_0\right) = 0$$

$$\boxed{\ddot{y} + \left(\frac{qB}{m}\right)^2 y + \left(\frac{qBv_0}{m} - \frac{qE}{m}\right) = 0} \qquad \ddot{y} + w^2 y + K = 0$$

Soln: $y = y_{homo.} + y_{particular.} = a\cos wt + b\sin wt + c$

$\dot{y} = -aw\sin wt + bw\cos wt + 0$

$\ddot{y} = -aw^2\cos wt - bw^2\sin wt$

$\ddot{y} + w^2 y + K = (-aw^2\cos wt - bw^2\sin wt) + w^2(a\cos wt + b\sin wt + c) + K = 0$

So $w^2 c + K = 0$ $\quad \boxed{c = -\frac{K}{w^2}}$

$\dot{y}(0) = 0 = -aw\sin 0 + bw\cdot 1 = 0$ $\quad \boxed{b = 0}$

$y(0) = 0 = a\cdot 1 + b\cdot 0 + -\frac{K}{w^2} = 0$ $\quad \boxed{a = K/w^2}$

(3.15 cont)

so the solution in y is

$$\boxed{y = \frac{K}{\omega^2}(\cos\omega t - 1) \qquad K = \frac{qBv_0}{m} - \frac{qE}{m} \qquad \omega = \frac{qB}{m}}$$

now back to x

$$\dot{x} = \frac{qB}{m} y + v_0 = \omega y + v_0 = \frac{\omega K}{\omega^2}(\cos\omega t - 1) + v_0$$

$$\int_{x_0}^{x} dx = \int_{t=0}^{t} \left[\frac{\omega K}{\omega^2}(\cos\omega t - 1) + v_0\right] dt$$

$$x - x_0 = \frac{K}{\omega}\left[\frac{\sin\omega t}{\omega} - t\right] + v_0 t \Big|_0^t \qquad x_0 = 0$$

so

$$\boxed{x = \frac{K}{\omega^2}\sin\omega t + \left[v_0 - \frac{K}{\omega}\right]t}$$

Summary

$$\left[\begin{array}{ll} x = \frac{K}{\omega^2}\sin\omega t + \left[v_0 - \frac{K}{\omega}\right]t & K = \frac{qBv_0}{m} - \frac{qE}{m} \\ y = \frac{K}{\omega^2}(\cos\omega t - 1) & \omega = \frac{qB}{m} \\ z = 0 & \end{array}\right]$$

matches the cycloid equations:

$x = a\sin\omega t + bt$

$y = c(1 - \cos\omega t)$

$z = 0$

$\boxed{3.16}$

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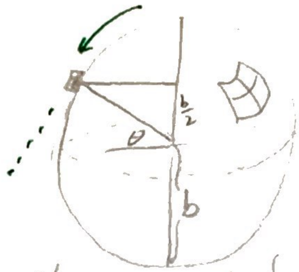

← starting point @ $h_0 = b/2$

when does the particle leave the sphere?

Energy $= \frac{1}{2}mv^2 + mgh$

Use Start: $E_{TOT} = E_{t=0} = 0 + mg\left(\frac{b}{2}\right) = \frac{mgb}{2}$

So $\frac{1}{2}mv^2 + mgh = \frac{mgb}{2}$

$$\underline{v(h) = \sqrt{g(b-2h)}}$$

Now restraint force $\vec{R} = -\vec{a}m$ at point where it leaves surface

$$\vec{a}_{centrip'l} = -\frac{v^2}{r} = -\frac{g(b-2h)}{b}$$

$$\vec{R} = mg\sin\theta\,\hat{r} = mg\left(\frac{h}{b}\right)\hat{r}$$

$\vec{R} + \vec{a}m = 0$

So

$$-g\frac{(b-2h)}{b}m = mg\left(\frac{h}{b}\right)$$

$$1 - \frac{2h}{b} = \frac{h}{b}$$

$$1 - 3\frac{h}{b} = 0 \qquad \boxed{h = \frac{b}{3}}$$

✓

$\boxed{3.20.}$ Spherical Pendulum $l = 1m$ $\theta_0 = 30°$

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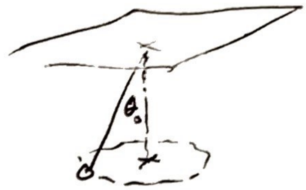
a) period of conical motion

b) period of osc θ about θ_0

c) angle of precession $\Delta\varphi$



a. rate of circular (conical motion) is

$\dot{\varphi}_0 = \sqrt{\frac{g}{l}\sec 30} = \sqrt{\frac{9.8 m/s^2}{1m}\frac{1}{\cos 30}} = 3.36/sec = \omega$ conical.

$T = \frac{2\pi \ R/cycle}{\omega \ R/sec} = \frac{cycle}{sec}$ $\quad \frac{2\pi \ cycle/sec}{3.36 \ R/sec} = 2\times3.14 = \frac{6.28}{3.36}$ sec/cycle ≈ 2

$\frac{2\pi}{\dot{\varphi}_0} = \frac{2\pi}{\sqrt{\frac{g}{l}\sec\theta_0}} = 1.87$ sec

b. $T_1 = 2\pi\sqrt{\frac{l}{gb}} = 2\pi\sqrt{\frac{1m}{(9.8)\left(\frac{3\sqrt{3}}{2}+\frac{6\sqrt{3}}{3}\right)\frac{?}{5}}} = \underline{1.036}$ sec ✓

c. $\Delta\varphi \approx \frac{3\pi}{8}\frac{\rho^2}{l^2} = \frac{3\pi}{8}\left(\frac{l^2\sin^2\theta_0}{l^2}\right) = \frac{3\pi}{8}\sin^2 30° = \frac{3\pi}{32} \sim .295$ ✓