Lagrangian Mechanics

9.1 Find the O.D.E's of motion of a projectile in a uniform gravitational field (No air resistance)

5/5

In cartesian coordinates

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$V = -mgz$$

Lagranges function $L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

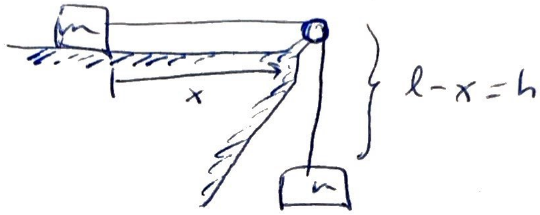
use

$$\dot{\left(\frac{\partial L}{\partial \dot{x}}\right)} = \frac{\partial L}{\partial x} \quad ; \quad \dot{\left(\frac{\partial L}{\partial \dot{y}}\right)} = \frac{\partial L}{\partial y} \quad ; \quad \dot{\left(\frac{\partial L}{\partial \dot{z}}\right)} = \frac{\partial L}{\partial z}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \, , \; \frac{\partial L}{\partial x} = 0 \quad ; \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y} \, , \; \frac{\partial L}{\partial y} = 0 \; ; \quad \frac{\partial L}{\partial \dot{z}} = m\dot{z} \, , \quad \frac{\partial L}{\partial z} = -mg$$

$$\Rightarrow \boxed{m\ddot{x} = 0 \, , \quad m\ddot{y} = 0 \quad m\ddot{z} = -mg}$$

9.3



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a) Find the acceleration of the system if a light cord.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{x}^2$$

$$V = -mgx$$

$$L = T - V = m\dot{x}^2 - mgx$$

$$\left(\frac{\partial L}{\partial \dot{x}}\right)^{\cdot} = \frac{\partial L}{\partial x} \Rightarrow 2m\ddot{x} = +mg$$

so $\boxed{\ddot{x} = \frac{g}{2}}$

b) $\ddot{x} = ?$ if cord weighs m'.

mass / length $= m'/l$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m'\dot{x}^2 + \frac{1}{2}m\dot{x}^2 = m\dot{x}^2 + \frac{m'}{2}\dot{x}^2 = \dot{x}^2(m + m'/2)$$

$$V = -mgx - \left(\frac{m'}{l}\right)(h)gx$$

$$L = m\dot{x}^2 + \frac{m'}{2}\dot{x}^2 + gx\left(-m - \frac{m'}{l}h\right)$$

$$\frac{\partial L}{\partial \dot{x}} = 2\left(m + \frac{m'}{2}\right)\dot{x}, \quad \frac{\partial L}{\partial x} = +mg + m'g\frac{h}{l}$$

$$\left(\frac{\partial L}{\partial \dot{x}}\right)^{\cdot} = 2\left(m + \frac{m'}{2}\right)\ddot{x} \equiv \frac{\partial L}{\partial x} = mg + \frac{m'gh}{l} \Rightarrow \boxed{\ddot{x} = \frac{g\left(m + m'\left(\frac{h}{l}\right)\right)}{2\left(m + \frac{m'}{2}\right)}}$$

9.5 Find the general QE. of motion for particle in spherical coordinates. use $v^2 = v_r^2 + v_\theta^2 + v_\varphi^2 = \dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\varphi^2$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left((h_r\dot r)^2 + (h_\theta\dot\theta)^2 + (h_\varphi\dot\varphi)^2\right)$$

$h_r = 1$
$h_\theta = r$
$h_\varphi = r\sin\theta$

$\frac{\partial T}{\partial \dot r} = m\dot r$; $\frac{\partial T}{\partial r}\left(\frac{1}{2}m2r\dot\theta^2 + \frac{1}{2}m2r\sin^2\theta\,\dot\varphi^2\right)$; $Q_r = m h_r a_r = F_r h_r$

$\frac{\partial T}{\partial \dot\theta} = mr^2\dot\theta$; $\frac{\partial T}{\partial \theta} = \frac{1}{2}mr^2 2\sin\theta\cos\theta\,\dot\varphi^2$; $Q_\theta = m h_\theta a_\theta = F_\theta h_\theta$

$\frac{\partial T}{\partial \dot\varphi} = m\dot\varphi r^2\sin^2\theta$; $\frac{\partial T}{\partial \varphi} = 0$; $Q_\varphi = m h_\varphi a_\varphi = F_\varphi h_\varphi$

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$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_k}\right) = Q_k + \frac{\partial T}{\partial q_k}$$

$\Rightarrow$

$$m\ddot r = mr\dot\theta^2 + mr\sin^2\theta\,\dot\varphi^2 + ma_r$$

$$m(r^2\ddot\theta + 2r\dot r\dot\theta) = mr^2\sin\theta\cos\theta\,\dot\varphi^2 + mra_\theta$$

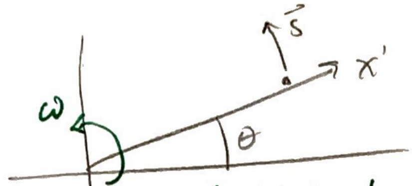
$$m\dot\varphi r^2 2\sin\theta\cos\theta\dot\theta + m\dot\varphi\, 2r\dot r\sin^2\theta + m\ddot\varphi\, r^2\sin^2\theta = 0 + m(r\sin\theta)a_\varphi$$

or

$$\begin{bmatrix} a_r = \ddot r - r\dot\theta^2 - r\sin^2\theta\,\dot\varphi^2 \\ a_\theta = \ddot\theta r + 2\dot r\dot\theta - r\sin\theta\cos\theta\,\dot\varphi^2 \\ a_\varphi = \ddot\varphi r^2\sin\theta + 2\dot\varphi\dot r\sin\theta + 2\dot\varphi r\dot\theta\cos\theta \end{bmatrix}$$

9.8

Lagrangian Mechanics Cont.



The plane starts to incline at rate ω = constant. $\theta = 0$ @ $t = 0$. Find motion of partile

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$$V = mg \sin\theta \, x'$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(v_T^2 + v_{\parallel}^2\right)$$

$$L = \frac{1}{2} m \left(\dot{x}'^2 + x'^2 \omega^2\right) - mg \, x' \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) = \frac{\partial L}{\partial x'}$$

$$\frac{\partial L}{\partial \dot{x}'} = m\dot{x}' \qquad \frac{\partial L}{\partial x'} = m x' \omega^2 - mg \sin\omega t$$

$$\dot{\left(\frac{\partial L}{\partial \dot{x}'}\right)} = m\ddot{x}'$$

so

$$\boxed{m\ddot{x}' = m x' \omega^2 - mg \sin\omega t} \qquad \underline{\ddot{x}' - \omega^2 x' = -g \sin\omega t}$$ ✓

the solution is of the form

$$x(t) = a e^{-\omega t} + b e^{+\omega t} + c \sin\omega t \qquad \boxed{c = \frac{g}{2\omega^2}}$$

plugging in initial condition $\dot{x}(0) = 0$, $x(0) = x_0$

$$\Rightarrow \quad x_0 = a + b$$

$$-\omega a + b\omega = \frac{-g}{2\omega^2}$$

$$\boxed{a = \frac{x_0}{2} + \frac{g}{4\omega^2}}$$

$$\boxed{b = \frac{x_0}{2} - \frac{g}{4\omega^2}}$$

So the motion equation is

$$\boxed{x(t) = \left(\frac{x_0}{2} + \frac{g}{4\omega^2}\right) e^{-\omega t} + \left(\frac{x_0}{2} - \frac{g}{4\omega^2}\right) e^{\omega t} + \frac{g}{2\omega^2} \sin\omega t}$$ ✓

good

$\boxed{9.9}$ Show that Lagrange's method automatically yields equations in rotating system.

Use $\vec{v} = \hat{\imath}(\dot{x} - \omega y) + \hat{\jmath}(\dot{y} + \omega x)$ and $T = \frac{1}{2} m \vec{v} \cdot \vec{v}$

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m \left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 \right]$$

10

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} + Q_x$$

$\frac{\partial V}{\partial \dot{x}} = 0$

$$\frac{d}{dt}\left(m(\dot{x} - \omega y) \right) = (\dot{y}\omega + \omega^2 x) m - V_x + Q_x$$

$$m\ddot{x} - \dot{\omega} y m - m\dot{y}\omega = m\dot{y}\omega + m\omega^2 x - V_x + Q_x$$

let $Q_x = 0$

$$\boxed{\ddot{x} - \dot{\omega} y - 2\dot{y}\omega - \omega^2 x = \frac{F_x}{m}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}$$

$\frac{\partial V}{\partial \dot{y}} = 0$

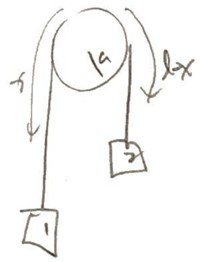
$$\frac{d}{dt}\left(\frac{2m}{2}(\dot{y} + \omega x)\right) = \frac{2m}{2}(\dot{x} - \omega y)\omega + F_y$$

$$m\ddot{y} + m\dot{\omega} x + m\dot{x}\omega = m\dot{x}\omega - m\omega^2 y + F_y$$

$$\boxed{\frac{F_y}{m} = \ddot{y} + \dot{\omega} x + 2\dot{x}\omega - \omega^2 y}$$

9.19 Write Hamitonian Functions & canonical eq. for...

10 (a) simple Atwood



$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}^2}{a^2}\right)$$

$$V = -x m_1 g - (l-x)m_2 g$$

$$H = T + V = \frac{1}{2}\dot{x}^2\left(m_1+m_2+\frac{I}{a^2}\right) - x m_1 g - (l-x)m_2 g$$

$$L = T - V = \frac{1}{2}\dot{x}^2\left(m_1+m_2+\frac{I}{a^2}\right) + x m_1 g + (l-x)m_2 g$$

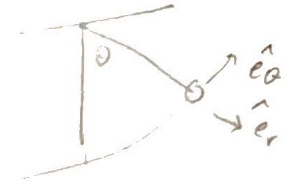
$$p_x = \frac{\partial L}{\partial \dot{x}} \Rightarrow p_x = \dot{x}\left(m_1+m_2+\frac{I}{a^2}\right)$$

So $$\boxed{H(p_x, x) = \frac{p_x^2}{2}\left(\frac{1}{m_1+m_2+I/a^2}\right) - m_1 g x - m_2 g(l-x)}$$

$$\frac{\partial H}{\partial x} = \boxed{-m_1 g + m_2 g = -\dot{p}_x} \qquad \frac{\partial H}{\partial p_x} = \boxed{\frac{p_x}{m_1+m_2+\frac{I}{a^2}} = \dot{x}}$$

9.19 (b)

Simple Pendulum



$$a_\theta = r\ddot{\theta} \qquad v_\theta = r\dot{\theta}$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (r^2 \dot{\theta}^2) \qquad V = -mgr(1-\cos\theta)$$

So

$$H = T + V = \frac{1}{2} m r^2 \dot{\theta}^2 - mgr(1-\cos\theta)$$

$$L = \frac{1}{2} m r^2 \dot{\theta}^2 + mgr(1-\cos\theta)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

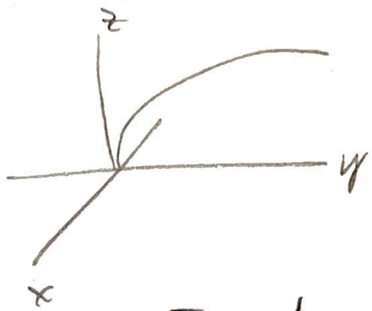
$$H(p_\theta, \theta) = \frac{1}{2} m r^2 \left(\frac{p_\theta}{m r^2}\right)^2 - mgr(1-\cos\theta)$$

$$\boxed{H = \frac{p_\theta^2}{2mr^2} - mgr(1-\cos\theta)}$$

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta = -(mgr(-\sin\theta)) = \boxed{mgr\sin\theta = -\dot{p}_\theta}$$ ✓

$$\frac{\partial H}{\partial p_\theta} = \boxed{\dot{\theta} = \frac{p_\theta}{mr^2}}$$ ✓

9.19 (e) projectile in uniform gravitational field.



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$V = mgz$$

$$H = T + V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$$

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$\boxed{H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz}$$

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k \qquad \frac{\partial H}{\partial p_k} = \dot{q}_k$$

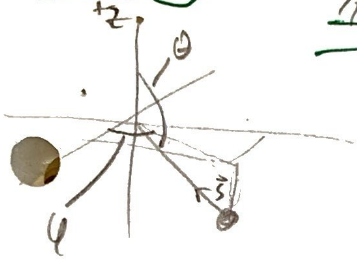
$$\Downarrow \qquad \Downarrow$$

$$\boxed{\begin{aligned} -\dot{p}_x &= 0 \\ -\dot{p}_y &= 0 \\ mg &= -\dot{p}_z \end{aligned}} \qquad \boxed{\begin{aligned} \frac{p_x}{m} &= \dot{x} \\ \frac{p_y}{m} &= \dot{y} \\ \frac{p_z}{m} &= \dot{z} \end{aligned}}$$

✓

$\boxed{9.19}$ (d)

spherical pendulum



$$\frac{ds}{dt} = \hat{e}_r h_r \frac{dr}{dt} + \hat{e}_\theta h_\theta \dot{\theta} + \hat{e}_\varphi h_\varphi \dot{\varphi} = v$$

so

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m\left(\cancel{\dot{r}^2}^{\,0} + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2\right)$$

$$V = -mgz = -mgr\sin\left(\theta - \frac{\pi}{2}\right) = mgr\cos\theta$$

but redefine $V=0$ point

$$V = +mgr\cos\theta - mgr = \underline{-mgr(1-\cos\theta)}$$

$$H = T+V = \frac{1}{2} m\left(r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) - mgr(1-\cos\theta)$$

$$L = T - V = \quad '' \quad + \quad ''$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = m r^2\dot{\theta} \qquad p_\varphi = m r^2 \sin^2\theta\, \dot{\varphi}$$

$$\boxed{H = \frac{1}{2m}\left(\frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) - mgr(1-\cos\theta)}$$

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k \qquad\qquad \frac{\partial H}{\partial p_k} = \dot{q}_k$$

$\Downarrow$

$$\boxed{\frac{1}{2m}\left(\frac{p_\theta^2}{r^3}(-2) + \frac{p_\varphi^2\,(-2)}{r^3\sin^2\theta}\right) - mg(1-\cos\theta) = -\dot{p}_r}$$

$$\boxed{-mgr\sin\theta = -\dot{p}_\theta}$$

$$\boxed{0 = -\dot{p}_\varphi}$$

$$\boxed{0 = \dot{r}}$$

$$\boxed{\frac{p_\theta}{mr^2} = \dot{\theta}}$$

$$\boxed{\frac{p_\varphi}{\cancel{2}mr^2 s^2\theta} = \dot{\varphi}}$$

THE LAGRANGIAN AND HAMILTONIAN FOR ELECTROMAGNETIC FORCES AND/OR NONINERTIAL COORDINATES

If the forces are electromagnetic forces on charged particles, then $m\ddot{\vec{r}} = q\vec{E} + q\,\vec{v}\times\vec{B}$. These forces are not conservative, in general.

Similarly, if the coordinate system is noninertial, the vector equation of motion contains non-conservative "fictitious" force terms, so that $m\ddot{\vec{r}} = \vec{F} - m\vec{A}_0 - 2m\,\vec{\omega}\times\dot{\vec{r}} - m\,\dot{\vec{\omega}}\times\vec{r} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$.

Since the forces are not conservative, there are no simple, scalar potentials which can be used to eliminate the forces entirely from Lagrange's equations. However, it turns out that we can use a combination of scalar and vector potentials in both problems to eliminate the forces, writing the Lagrangian in terms of these potentials alone.

In the electromagnetic case, we can write

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla}\times\vec{A},$$

where $\vec{A} = \vec{A}(\vec{r},t)$ is the vector potential, and $\phi = \phi(\vec{r},t)$ is the scalar potential.

We can use rectangular coordinates to explore the equations of motion, since Lagrange's equations are independent of coordinates--so we do the easiest coordinate system, but the result will be valid in any coordinates.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} = m\ddot{x}_i = F_i = qE_i + q(\vec{v}\times\vec{B})_i$$

$$= -q\frac{\partial\phi}{\partial x_i} - q\frac{\partial A_i}{\partial t} + q[\vec{v}\times(\vec{\nabla}\times\vec{A})]_i$$

$$\vec{v}\times(\vec{\nabla}\times\vec{A}) = \overbrace{\vec{\nabla}(\vec{v}\cdot\vec{A})} - (\vec{v}\cdot\vec{\nabla})\vec{A}\,,$$ where $\vec{\nabla}$ operates only on $\vec{A}$ in the first term on the right.

$$\Rightarrow [\vec{v}\times(\vec{\nabla}\times\vec{A})]_i = \sum_{j=1}^{3}\dot{x}_j\frac{\partial A_j}{\partial x_i} - \sum_{j=1}^{3}\dot{x}_j\frac{\partial A_i}{\partial x_j}.$$

Now $$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \sum_{j=1}^{3}\frac{\partial A_i}{\partial x_j}\dot{x}_j$$

$$\Rightarrow F_i = -q\frac{\partial\phi}{\partial x_i} - q\frac{\partial A_i}{\partial t} + q\sum_{i=1}^{3}\dot{x}_j\frac{\partial A_j}{\partial x_i} - q\sum_{i=1}\dot{x}_j\frac{\partial A_i}{\partial x_j}$$

$$= -q\frac{\partial\phi}{\partial x_i} - q\frac{dA_i}{dt} + q\sum_{i=1}^{3}\dot{x}_j\frac{\partial A_j}{\partial x_i}.$$

So if we let $\underline{L = T - q\phi + q\vec{v}\cdot\vec{A}}$, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 .$$

Then $\underline{p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i}$, or $\vec{p} = m\vec{v} + q\vec{A}$ (!)

$$H = \sum_{i=1}^{3} p_i\dot{x}_i - L = \vec{p}\cdot\vec{v} - T + q\phi - q\vec{v}\cdot\vec{A}$$

$$= \vec{p}\cdot\frac{\vec{p}-q\vec{A}}{m} - \frac{(\vec{p}-q\vec{A})^2}{2m} + q\phi - \frac{q}{m}(\vec{p}-q\vec{A})\cdot\vec{A}$$

$$= \frac{(\vec{p}-q\vec{A})^2}{2m} + q\phi = T + q\phi = E .$$

In the case of a noninertial coordinate system, again using rectangular coordinates for simplicity, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} = m\ddot{x}_i = F_i - mA_{0i} - 2m(\vec{\omega}\times\dot{\vec{r}})_i - m(\dot{\vec{\omega}}\times\vec{r})_i - m[\vec{\omega}\times(\vec{\omega}\times\vec{r})]_i$$

Let $F_i = -\frac{\partial V}{\partial x_i} + F_i'$, where F_i' is not conservative.

Then $$m\ddot{x}_i = F_i' - \frac{\partial}{\partial x_i}\left[V + m\vec{r}\cdot\vec{A}_0 + \frac{m}{2}\vec{r}\cdot(\vec{\omega}\times(\vec{\omega}\times\vec{r}))\right] - 2m(\vec{\omega}\times\dot{\vec{r}})_i - m(\dot{\vec{\omega}}\times\vec{r})_i$$

The Coriolis term makes $\vec{\omega}$ look like an analog of $\vec{B}$ in the electromagnetic case. So we look for a vector potential which satisfies $\vec{\omega} = \vec{\nabla}\times\vec{A}$. Since $\vec{\omega} = \vec{\omega}(t)$, we easily find $\vec{A} = \frac{1}{2}\vec{\omega}\times\vec{r}$.

So we try $\underline{L = T - V - m\vec{r}\cdot\vec{A}_0 - \frac{1}{2}m\vec{r}\cdot(\vec{\omega}\times(\vec{\omega}\times\vec{r})) + m\vec{v}\cdot\vec{\omega}\times\vec{r}}$.

This gives $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = F_i'$, as desired.

Now $\underline{p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + m(\vec{\omega}\times\vec{r})_i}$, or $\vec{p} = m\vec{v} + m(\vec{\omega}\times\vec{r})$.

So $\vec{p}$ is the momentum relative to coordinates which are not rotating.

Since $\vec{v} = \frac{\vec{p} - m\vec{\omega}\times\vec{r}}{m}$, $H = \vec{p}\cdot\vec{v} - L = \frac{p^2 - \vec{p}\cdot m(\vec{\omega}\times\vec{r})}{m} - \frac{(\vec{p}-m\vec{\omega}\times\vec{r})^2}{2m} + V + m\vec{r}\cdot\vec{A}_0 + \frac{1}{2}m\vec{r}\cdot(\vec{\omega}\times(\vec{\omega}\times\vec{r})) - (\vec{p}-m\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})$

$$\Rightarrow H = \frac{(p - m\vec{\omega}\times\vec{r})^2}{2m} + V + m\vec{r}\cdot\vec{A}_0 + \frac{1}{2}m\vec{r}\cdot(\vec{\omega}\times(\vec{\omega}\times\vec{r})) = T + V_{eff}.$$