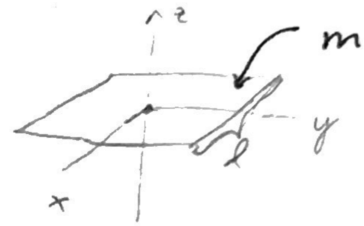
Rigid body in 3-D

8.1 Write $\mathbb{I}$ for plate



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$I_{xx} = \int (y^2+z^2)\,dm$ but $z=0$ $\qquad \rho = \frac{m}{\ell^2}$

$$= 2\int_0^{\ell/2} y^2(\ell\,dy\,\rho) = \frac{2}{3} y^3 \frac{\ell m}{\ell^2}\Big|_0^{\ell/2} = \frac{\ell^2 m}{12}$$

$I_{yy} = I_{xx} = \frac{\ell^2 m}{12}$ $\qquad I_{zz} = I_{xx} + I_{yy} = \frac{2\ell^2 m}{12} = \frac{\ell^2 m}{6}$

$$I_{yx} = I_{xy} = -\int xy\,dm = -\int_{-\ell/2}^{\ell/2}\int_{-\ell/2}^{\ell/2} yx\,dx\,dy\,\rho = -\rho\int_{-\ell/2}^{\ell/2} y\,dy\left[\frac{x^2}{2}\right]_{-\ell/2}^{\ell/2} = 0$$

So
$$\boxed{\mathbb{I} = \ell^2 m \begin{bmatrix} \frac{1}{12} & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}}$$ ✓

8.2 find $\mathbb{L}$ & T when it rotates about a diagonal:

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$$\vec{\omega} = \frac{\omega}{\sqrt{\ell^2+\ell^2}}\begin{bmatrix}\ell\\ \ell\\ 0\end{bmatrix} = \frac{\omega}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

same direction $\vec{\omega}$

$$\mathbb{L} = \mathbb{I}\,\omega = \frac{\ell^2 m}{\sqrt{2}\,12}\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\omega = \frac{m\ell^2\omega}{12\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$ ✓

$$T = \frac{1}{2}\omega^T\,\mathbb{L} = \frac{1}{2}\left[\frac{\omega}{\sqrt{2}}[1,1,0]\right]\begin{bmatrix}1\\1\\0\end{bmatrix}\frac{m\ell^2\omega}{12\sqrt{2}} = \frac{\omega^2\ell^2 m\,\cancel{2}}{12\cdot 2\,\cancel{2}}$$

$$\boxed{T = \frac{\omega^2\ell^2 m}{24}}$$ ✓

$\boxed{8.3}$

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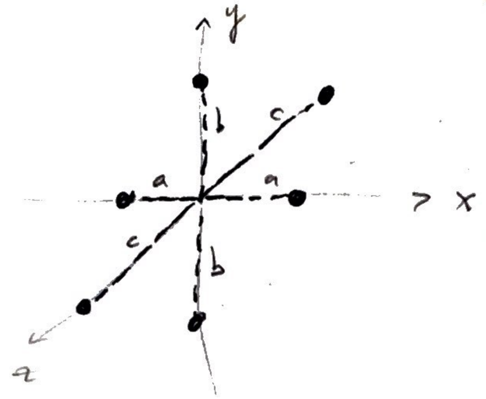


Show if coord. coincide with rods, they are principal axes, write down the inertia tensor.

By symmetry it can be shown body axes are principal axes. But the results of $\mathbb{I}$ will reveal it.

- $I_{xx} = \sum_i (y_i^2 + z_i^2)\, m_i = m[0^2 + 0^2 + (b^2) + (-b)^2 + c^2 + (-c)^2]$
  $= \underline{2m(b^2+c^2)}$

- $I_{yy} = \sum_i (x_i^2 + z_i^2)\, m_i = m[a^2 + a^2 + 0 + 0 + c^2 + c^2]$
  $= \underline{2m(a^2+c^2)}$

- similar for I_{zz}, ie $\underline{I_{zz} = 2m(b^2+a^2)}$

- $\underline{I_{xy} = I_{yx}} = -\sum x_i y_i m_i = -m[0+0+0+\cdots] = \underline{0}$

  similar $I_{yz}, I_{zy}, I_{xz}, I_{zx} = 0$

so $$\boxed{\mathbb{I} = 2m\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & b^2+a^2 \end{bmatrix}}$$ ✓

because this is a diagonal matrix, ie. $L_{xx}, L_{yy}, L_{zz} \neq 0$

its coincident with principal axes ✓

$\boxed{8.4}$ Find the $\vec{L}$ & T (KE) of 8.3 when rot. about axis $\overline{OP}$ where $P = (a, b, c)$ at speed ω

$$\omega = \omega \begin{bmatrix} a \\ b \\ c \end{bmatrix} \frac{1}{\sqrt{a^2+b^2+c^2}}$$

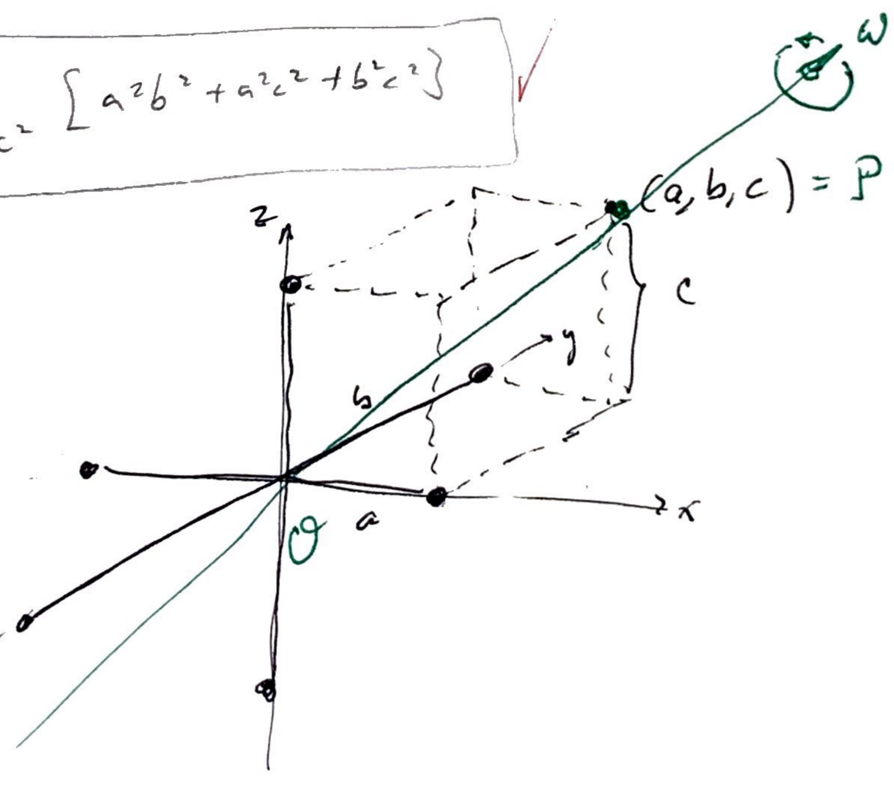
$$\mathbb{L} = \mathbb{I}\,\omega = \frac{2m\omega}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boxed{\mathbb{L} = \frac{2m\omega}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} a(b^2+c^2) \\ b(a^2+c^2) \\ c(a^2+b^2) \end{bmatrix}}$$

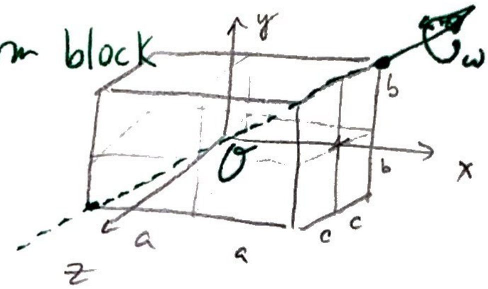
$$T = \frac{1}{2}\omega^T \mathbb{I} = \frac{1}{2}\frac{\omega}{N}[a, b, c] \begin{bmatrix} a(b^2+c^2) \\ b(a^2+c^2) \\ c(a^2+b^2) \end{bmatrix} \frac{2m\omega}{N}$$

$$= \frac{m\omega^2}{a^2+b^2+c^2}\left[a^2(b^2+c^2) + b^2(a^2+c^2) + c^2(a^2+b^2)\right]$$

$$\boxed{T = \frac{2m\omega^2}{a^2+b^2+c^2}\left[a^2b^2 + a^2c^2 + b^2c^2\right]}$$



8.5 uniform block



(a) Find $\mathbb{I}$ CM @ $(0,0,0)$

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- $I_{xx} = \int (y^2+z^2)\,dm = \rho\int y^2\,dy\,dx\,dz + \rho\int z^2\,dz\,dy\,dx$

$$= \rho\,\frac{y^3}{3}\Big|_{-b}^{b}\, x\Big|_{-a}^{a}\, z\Big|_{-c}^{c} + \rho\,\frac{z^3}{3}\Big|_{-c}^{c}\, y\Big|_{-b}^{b}\, x\Big|_{-a}^{a} \qquad \rho = \frac{m}{2a2b2c}$$

$$= \rho\left\{\frac{2}{3}b^3\,2a\,2c + \frac{2}{3}c^3\,2b\,2a\right\} = \frac{m}{3}\left[b^2+c^2\right]$$

- I_{yy} is cyclically similar to I_{xx} (exchange $y\to z$, $z\to x$ or $b\to c$, $c\to a$)

so $I_{yy} = \frac{m}{3}\left[c^2+a^2\right]$

- similarly $I_{zz} = \frac{m}{3}\left[a^2+b^2\right]$

- $I_{xy} = -\rho\int xy\,dx\,dy\,dz = \frac{x^2}{2}\Big|_{-a}^{a}\,\frac{y^2}{2}\Big|_{-b}^{b}\,z\Big|_{-c}^{c}$ 2nd order, symmetry $\Rightarrow = 0$

- similar for I_{yz}, I_{xz}

so

$$\boxed{\mathbb{I} = \frac{m}{3}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}}$$

8.5 cont)

if $\hat{n} = OP \qquad P = (a, b, c)$

then $\quad \omega = \dfrac{\omega}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\mathbb{L} = \frac{m}{3} \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & b^2+a^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \frac{\omega}{\sqrt{a^2+b^2+c^2}}$$

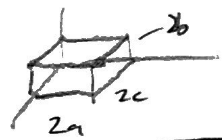
$$\boxed{\mathbb{L} = \frac{\omega m}{3\sqrt{a^2+b^2+c^2}} \begin{bmatrix} a(b^2+c^2) \\ b(a^2+c^2) \\ c(b^2+a^2) \end{bmatrix}} \checkmark$$

$$T = \frac{1}{2} \omega^T \mathbb{L} = \frac{1}{2}\left(\frac{\omega}{\sqrt{\ }}\right)\left(\frac{\omega m}{3\sqrt{\ }}\right) [a, b, c] \begin{bmatrix} a(\quad) \\ b(\quad) \\ c(\quad) \end{bmatrix}$$

$$= \frac{\omega^2 m}{2 \cdot 3(a^2+b^2+c^2)} \left\{a^2(b^2+c^2) + b^2(a^2+c^2) + c^2(b^2+a^2)\right\}$$

$$\boxed{T = \frac{\omega^2 m}{3(a^2+b^2+c^2)} \left[a^2b^2 + b^2c^2 + c^2a^2\right]} \checkmark$$

(b) for origin @ a corner:



$$I_{xx} = \int y^2 + z^2 \rho \, dx \, dy \, dz$$

$$= \rho \frac{y^3}{3}\Big|_0^{2b} z\Big|_0^{2c} x\Big|_0^{2a} + \rho \frac{z^3}{3}\Big|_0^{2c} x\Big|_0^{2a} y\Big|_0^{2b}$$

$$= \left(\overset{=\rho}{\frac{m}{8abc}}\right)\left[\frac{8b^3\, 2c\, 2a}{3} + \frac{8c^3\, 2a\, 2b}{3}\right] = \frac{4m}{3}\left[c^2 + b^2\right]$$

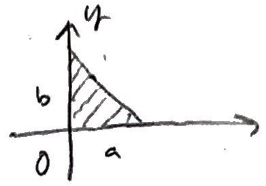
similarly $\quad I_{yy} = \frac{4}{3} m\left[a^2+c^2\right], \quad I_{zz} = \frac{4}{3} m\left[a^2+b^2\right]$

$$I_{xy} = -\rho\int xy \, dx\, dy\, dz = -\rho \frac{x^2}{2}\Big|_0^{2a} \frac{y^2}{2}\Big|_0^{2b} z\Big|_0^{2c} = \left(\frac{-m}{8abc}\right)\frac{4a^2 4b^2\, 2c}{4} = -mab$$

similarly for $\underline{I_{yz} = -mbc}, \quad I_{xz} = -mac$.

$$\therefore \boxed{\mathbb{I} = m \begin{bmatrix} \frac{4}{3}(b^2+c^2) & -ab & -ac \\ -ab & \frac{4}{3}(a^2+c^2) & -bc \\ -ac & -bc & \frac{4}{3}(a^2+b^2) \end{bmatrix}} \checkmark$$

8.6 (a) Find $\mathbb{I}$ for a triangle



$y = -\frac{b}{a}x + b$

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- $I_{xx} = \rho \int (y^2 + z^2)\, dx\, dy$ but $z = 0$ $\quad I_{xx} = \rho \int y^2\, dx\, dy = \rho \int_0^a \int_0^{-\frac{b}{a}x+b} y^2\, dy\, dx$

$= \rho \int_0^a \frac{(-\frac{b}{a}x+b)^3}{3}\, dx$ let $u = -\frac{b}{a}x + b$ $\quad du = -\frac{b}{a}\, dx$

$= \rho\left(\frac{a}{b}\right) \int_0^b \frac{u^3}{3}\, du = \frac{\rho a}{b} \frac{b^4}{3 \cdot 4} = \left(\frac{m}{\frac{1}{2}ab}\right) \frac{a b^3}{12} = \frac{1}{6} m b^2$

- $I_{yy} = \rho \int (x^2 + 0^2) \rho\, dx\, dy = \rho \int_0^b \int_0^{(y-b)a/b} x^2\, dx\, dy = \rho \frac{a^3}{b^3} \int_0 \frac{(y-b)^3}{3}\, dy$

$= \left(\frac{m}{\frac{1}{2}ab}\right) \frac{1}{3} \frac{a^3}{b^3} \frac{b^4}{4} = \frac{m a^3}{6a} = \frac{m a^2}{6}$

(b)

- by $\perp$ axis theorem $\quad I_{zz} = I_{xx} + I_{yy} = \frac{m}{6}(a^2 + b^2)$

- $I_{xz}, I_{yz} = 0 \quad I_{xy} = -\rho \int xy\, dx\, dy = \rho \int_0^a \int_0^{-\frac{bx}{a}+b} x y\, dy\, dx$

$= \frac{-m}{(\frac{1}{2}ab)} \int_0^a x\, dx \left[\frac{(-\frac{bx}{a}+b)^2}{2}\right] = -\int_0^a x\left[\frac{b^2}{a^2}x^2 + 2\frac{b^2}{a}x + b^2\right] dx$

$= -\left(\frac{b^2}{a^2}\frac{a^4}{4} - \frac{2}{3}\frac{b^2 a^3}{a} + \frac{b^2 a^2}{2}\right)\rho = -b^2 a^2 \left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right]\rho = -b^2 a^2 \left(\frac{1}{12}\right)\rho = \frac{-2m}{ab}\left(\frac{b^2 a^2}{12}\right)$

$I_{xy} = -\frac{mab}{6}$

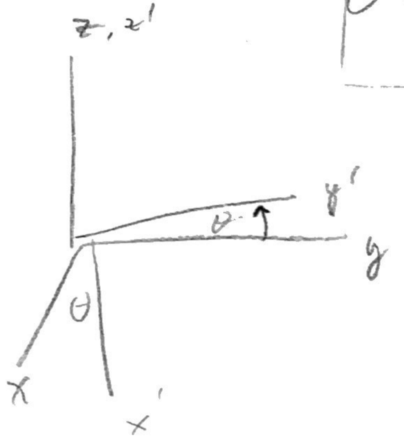
(c)

$$\boxed{\mathbb{I} = \frac{m}{6} \begin{bmatrix} b^2 & -ab/2 & 0 \\ -ab/2 & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}}$$

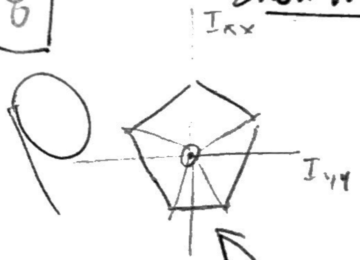
(8.6 cont) (b) Find principle axii (eigenvectors of $\mathbb{I}$)
if non-lamina object

For lamina (plane) $\left.\right\}$ $\tan 2\theta = \dfrac{2 I_{xy}}{I_{xx} - I_{yy}} = \dfrac{-2ab}{a^2 - b^2}$

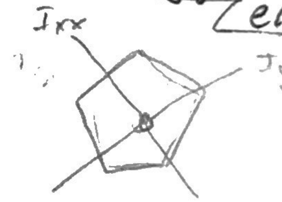
$$\boxed{\theta = \frac{\tan^{-1}}{2}\left(\frac{2ab}{b^2 - a^2}\right)}$$



8.8 Show momental ellipsoid of lamina regular polygon is an ellipsoid.



if we turn $\dfrac{\frac{2\pi}{n}}{2}$ Radians



here I_{xx} is symetrized (a principal axis)

now if the laminal polygon is uniform mass, the c.m. is coincident with the geometric center.

Turning $\dfrac{\frac{2\pi}{n}}{2}$ about this c.m., now I_{yy} is the "balanced" I. Another $\dfrac{\frac{2\pi}{n}}{2}$ yields the same as the initial condition, back to I_{xx}.

i.e. I_{xx} & I_{yy} are cyclical exchangable.

over ⇒

So in $x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} = 1$ is cyclical in x^2, y^2 ...

components

So initial axes yield,

$$x^2 I_{0xx} + y^2 I_{0yy} + z^2 I_{zz} = 1$$

1st trm

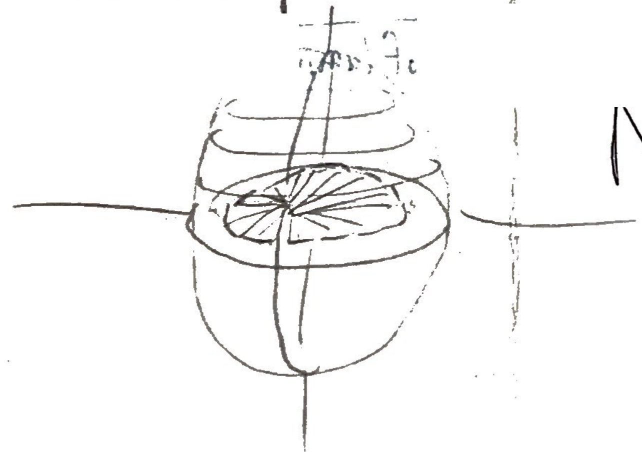
$$x^2 I_{0yy} + y^2 I_{0xx} + z^2 I_{zz} = 1$$

2nd

$$x^2 I_{0xx} + y^2 I_{0yy} + z^2 I_{zz} = 1$$

for a circle $ax^2 + by^2 \Rightarrow a = b$ *

so ellipsoid is circular in x, y plane.



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let $I_{zz} = I_{xx} + I_{yy}$ by $\perp$ axis

$$x^2(I_{xx}) + y^2 I_{yy} + z^2 I_{zz} = 1$$

$$x^2(I_{zz} - I_{yy}) + y^2 I_{yy} + z^2 I_{zz} = 1$$

$$x^2 I_{zz} - x^2 I_{yy} + y^2 I_{yy} + z^2 I_{zz} = 1$$

$$(-x^2 + y^2) I_{yy} + (z^2 + x^2) I_{zz} = 1$$

let $z = 0$ to examine a plane section

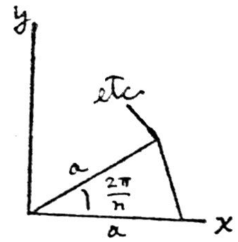
$$(-x^2 + y^2) I_{yy} + x^2 I_{zz} = 1$$

$$x^2(I_{zz} - I_{yy}) + y^2 I_{yy} = 1$$

Excellent work

please see the key for a good math proof.

8.8) An n-gon is composed of n triangles, each with two sides of length a and angle between them $\frac{2\pi}{n}$.



Take $z \perp$ plane of lamina, x along a side of a triangle, as shown.

There will be $\frac{n}{2}$ triangles for $y>0$ and $n/2$ for $y<0$.

$\Rightarrow$ $I_{xy} = 0$. So we have principal axes.

$I_{zz} = I_{xx} + I_{yy}$

The n-gon has an n-fold axis of symmetry along z.
So if we rotate axes by $\frac{2\pi k}{n}$, $I_{x'x'} = I_{xx}$ and $I_{y'y'} = I_{yy}$.
$\Rightarrow$ Momental ellipsoid has an n-fold axis of symmetry along z. The only way an ellipsoid can have an n-fold axis with $n>2$ is if it is an ellipsoid of revolution.

Proof: For our axes the momental ellipsoid is given by

$$x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} = 1.$$

$$\vec{r}^{\,T} \underline{\underline{I}}\, \vec{r} = 1$$

We can rotate axes by $\theta = \frac{2\pi k}{n}$ by the orthogonal transformation

$$\underline{\underline{S}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ (See p. 19 of text.)

$\vec{r}\,' = \underline{\underline{S}}\,\vec{r}$, $\underline{\underline{S}}^{-1} = \underline{\underline{S}}^T$, $\vec{r}\,'^T = \vec{r}^{\,T}\underline{\underline{S}}^T \Rightarrow \vec{r} = \underline{\underline{S}}^T \vec{r}\,'$
$\vec{r}^{\,T} = \vec{r}\,'^T \underline{\underline{S}}$

$\Rightarrow$ $\vec{r}\,'^T \underline{\underline{S}}\, \underline{\underline{I}}\, \underline{\underline{S}}^T \vec{r}\,' = 1$

$\Rightarrow$ $\underline{\underline{I}}' = \underline{\underline{S}}\,\underline{\underline{I}}\,\underline{\underline{S}}^T = \underline{\underline{I}}$ since we are rotating about an n-fold axis of symmetry.

(8.8 continued)

$$\underline{\underline{I}}\,\underline{\underline{S}}^T = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{xx}\cos\theta & -I_{xx}\sin\theta & 0 \\ I_{yy}\sin\theta & I_{yy}\cos\theta & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

$$\underline{\underline{I}}' = \underline{\underline{S}}\,\underline{\underline{I}}\,\underline{\underline{S}}^T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} I_{xx}\cos\theta & -I_{xx}\sin\theta & 0 \\ I_{yy}\sin\theta & I_{yy}\cos\theta & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} I_{xx}\cos^2\theta + I_{yy}\sin^2\theta & (I_{yy}-I_{xx})\sin\theta\cos\theta & 0 \\ (I_{yy}-I_{xx})\sin\theta\cos\theta & I_{xx}\sin^2\theta + I_{yy}\cos^2\theta & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

$$\underline{\underline{I}}' = \underline{\underline{I}} \Rightarrow (I_{yy} - I_{xx})\sin\theta\cos\theta = 0$$

$$\Rightarrow I_{xx} = I_{yy} \quad \text{or} \quad \sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta = 0$$

If $2\theta \neq m\pi$, then <u>$I_{xx} = I_{yy}$ and the ellipsoid is one of revolution.</u>

But $\theta = \dfrac{2\pi k}{n}$, $k = 1, .., n$.

$$\Rightarrow 2\theta = \frac{4k}{n}\pi$$

$n \geq 3$. So if $n = 3$, $k=1 \Rightarrow I_{xx} = I_{yy}$.

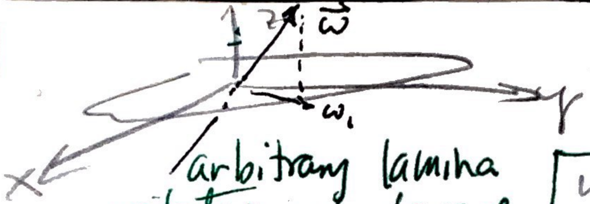
if $n = 4$ we have a square and $I_{xx} = I_{yy}$ by symmetry.

if $n > 4$ but not a multiple of 4, $2\theta \neq m\pi$ for some symmetry rotations.

if n is a multiple of 4, the four quadrants have identical mass distributions, so $I_{xx} = I_{yy}$ by symmetry.

<u>$\therefore$ $I_{xx} = I_{yy}$ for all n-gons, $n \geq 3$. $\Rightarrow$ Momental ellipsoid is one of revolution.</u>

8.10



arbitrary lamina rotates w/o torque

$\omega_1^2 = \omega_x^2 + \omega_y^2$

show ω_1 = const. but $\omega_z \neq$ const.

we need to show ω_1^2 = const.

or $\omega_x^2 + \omega_y^2$ = const.

or it's derivative = 0 $\quad 2\omega_x \dot{\omega}_x + 2\omega_y \dot{\omega}_y = 0$

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or

$$\frac{\dot{\omega}_x}{\dot{\omega}_y} = \frac{-\omega_y}{\omega_x}$$

from Euler eq.

Because no torque,

$$N_x = I_x \dot{\omega}_x + \omega_y \omega_z (I_z - I_y) = 0$$
$$N_y = I_y \dot{\omega}_y + \omega_z \omega_x (I_x - I_z) = 0$$

putting in $I_z = I_x + I_y$ in to the N_x

$\Rightarrow \quad I_x \dot{\omega}_x + \omega_y \omega_z I_x = 0 \quad \Rightarrow \quad \dot{\omega}_x = -\omega_y \omega_z$

N_y eq. $\Rightarrow \quad I_y \dot{\omega}_y + \omega_z \omega_x (-I_y) = 0 \quad \Rightarrow \quad \dot{\omega}_y = +\omega_z \omega_x$

divide

$$\frac{\dot{\omega}_x = -\omega_y \omega_z}{\dot{\omega}_y = \omega_z \omega_x}$$

or

$$\frac{\dot{\omega}_x}{\dot{\omega}_y} = \frac{-\omega_y}{\omega_x}$$

N_z := $I_z \dot{\omega}_z + \omega_x \omega_y (I_y - I_x) = 0$

for $I_y \neq I_x$

$= I_z \dot{\omega}_z + \underbrace{\omega_x \omega_y}_{\text{not const}} (I_y - I_x) = 0$

if not on principal axis

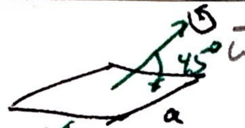
precession

so $I_z \dot{\omega}_z$ = varying

$\Rightarrow \dot{\omega}_z \neq$ const

unless $I_y = I_y$ then $I_z \dot{\omega}_z + 0 = 0$ $\quad \underline{\omega_z = \text{const}}$

$\boxed{8.11.}$



$\vec{\omega}$ about $\hat{x}$ $\qquad$ $\hat{z}$ about I

(a) $\Omega = \omega\left[\frac{I_s}{I} - 1\right]\cos\alpha$; $\dot{\varphi} = \omega\left[1+\left(\left(\frac{I_s}{I}\right)^2-1\right)\cos^2\alpha\right]^{\frac{1}{2}}$

Find period of precession

Thin plate $I_s = \frac{1}{6}a^2$, $I = \frac{a^2}{12}$

← period

so $\underline{\underline{T_\Omega}} = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega}\frac{1}{\frac{1}{\sqrt{2}}\left[\frac{\frac{1}{6}}{1/12}-1\right]} = \underline{\underline{\frac{2\pi\sqrt{2}}{\omega}}}$ ✓

$\underline{\underline{T_{\dot\varphi}}} = \frac{2\pi}{\dot\varphi} = \frac{2\pi}{\omega}\frac{1}{\left[1+\left[\left(\frac{\frac{1}{6}a^2}{\frac{a^2}{12}}\right)^2-1\right]\frac{1}{2}\right]^{\frac{1}{2}}} = \frac{2\pi}{\omega}\cdot\frac{1}{\left[5/2\right]^{1/2}} = \underline{\underline{\frac{2\pi}{\omega}\left(\frac{2}{5}\right)^{\frac{1}{2}}}}$ ✓

(b) ~~Thick~~ plate $I_s = \frac{a^2}{6}$, $I = m\frac{(a^2+c^2)}{12}$ but $c = \frac{a}{4}$

So $I = \frac{15\cdot}{16\cdot12} = \frac{5}{16\cdot4} = \frac{5}{64}a^2$

$\underline{\underline{T_\Omega}} = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega}\frac{1}{\left[\frac{\frac{a^2}{6}}{5/64}-1\right]\frac{1}{\sqrt{2}}} = \frac{2\pi\sqrt{2}}{\omega}\frac{30}{64-30} = \underline{\frac{2\pi\sqrt{2}}{\omega}\frac{15}{17}}$ ✗ $\frac{34\sqrt{2}\pi}{15\omega}$

$T_\varphi = \frac{2\pi}{\dot\varphi} = \frac{2\pi}{\omega}\frac{1}{\left[1+\left\{\left(\frac{\frac{a^2}{6}}{5/64\,a^2}\right)^2-1\right\}\frac{1}{2}\right]^{\frac{1}{2}}} = \frac{2\pi}{\omega}\frac{1}{\left[1+\left[\left(\frac{64}{30}\right)^2-1\right]\right]^{\frac{1}{2}}}$

$\frac{4096-900}{2\cdot300} + 2600$

$= \underline{\underline{\frac{2\pi}{\omega}\left[\frac{600}{3796}\right]^{\frac{1}{2}}}}$ ✗ $\frac{1.33\pi}{\omega}$

8.13 Find the angle between $\vec{\omega}$ & $\vec{L}$ for two cases in #11

$\measuredangle = \alpha - \theta$

$\dot{\varphi} = \omega \frac{\sin\alpha}{\sin\theta}$ $\quad \sin^{-1}\left(\frac{\omega}{\dot{\varphi}} \sin\alpha\right) = \theta$

10

$\measuredangle = 45° - \tan^{-1}\left(\frac{I}{I_s} \tan\alpha\right)$

a) $\measuredangle = 45° - \tan^{-1}\left(\frac{\frac{a^2}{12}}{\frac{a^2}{6}}(1)\right) = 45° - \tan^{-1}\left(\frac{1}{2}\right) = 18.4°$

b) $\measuredangle = 45° - \tan^{-1}\left(\frac{\frac{a^2}{\cancel{12}} + c^2}{\frac{a^2}{6}}(1)\right)$ $= 45° - \tan^{-1}\left(\frac{17}{32}\right) = 17.02°$

$= 45° - \tan^{-1}\left(\frac{1}{2} + \frac{1}{16}\right)$

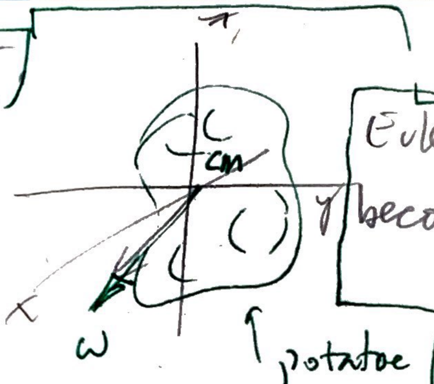
$\measuredangle = 45° - \tan^{-1}\left(\frac{a^2 + c^2}{2a^2}\right) = 45° - \tan^{-1}\left(\frac{1}{2} + \frac{c^2}{2a^2}\right)$

Demonstrate the intermediate axis theorm

8.15

let $I_z >> I_y > I_x$

10



No Torques

Eulers become

$\dot{\omega}_x = -\omega_y \omega_z (I_z - I_y)/I_x$ — I_{middle}

$\dot{\omega}_y = -\omega_z \omega_x (I_x - I_z)/I_y$ — I_{big}

$\dot{\omega}_z = -\omega_x \omega_y (I_y - I_x)/I_z$ — I_{small}

if $\vec{\omega} \parallel \hat{x}$ and displaced alittle, $\Delta\omega$

then $\dot{\omega}_x = -\overbrace{(\omega_y + \Delta\omega_y)(\omega_z + \Delta\omega_z)}^{small} I_m \approx 0$ stable (ω_x won't change drastically)

$\dot{\omega}_y = -(\omega_z + \Delta\omega_z)(\omega_x + \Delta\omega_x) I_b$

$\dot{\omega}_z = -(\omega_x + \Delta\omega_x)(\omega_y + \Delta\omega_y) I_s$

$\vec{\omega} \parallel \hat{y}$

$\dot{\omega}_x = -(\omega_y + \Delta\omega_y)(\omega_z + \Delta\omega_z) I_m$ small

$\dot{\omega}_y = -(\omega_z + \Delta\omega_z)(\omega_x + \Delta\omega_x) I_b$ — Big unstable.

$\dot{\omega}_z = -($ " " $)$

please see the key

(oh well)

8.15) Euler's equations for zero torque:

$$I_i \dot{\omega}_i + \omega_j \omega_k (I_k - I_j) = 0 \qquad (i\,j\,k) = \text{cyclic permutation of } (x\,y\,z)$$

$$(I_x = I_{xx},\ I_y = I_{yy},\ I_z = I_{zz})$$

$$\Rightarrow \dot{\omega}_i = -\omega_j \omega_k \frac{(I_k - I_j)}{I_i}.$$

Three solutions to these equations are

$$\omega_i = \text{constant}, \quad \omega_j = \omega_k = 0.$$

To examine stability, assume we are very close to one of these solutions, so $\omega_i = \omega_0 + \epsilon_i$, $\omega_j = \epsilon_j$, $\omega_k = \epsilon_k$. The ϵ's are small, so we can neglect ϵ^2 terms.

For simplicity of notation take $i\,j\,k = x\,y\,z$ now.

$$\Rightarrow \dot{\omega}_x = \dot{\epsilon}_x = -\epsilon_y \epsilon_z \left(\frac{I_z - I_y}{I_x}\right) \cong 0$$

$$\dot{\omega}_y = \dot{\epsilon}_y \cong -\epsilon_z \omega_0 \left(\frac{I_x - I_z}{I_y}\right)$$

$$\dot{\omega}_z = \dot{\epsilon}_z \cong -\omega_0 \epsilon_y \left(\frac{I_y - I_x}{I_z}\right)$$

Solve the last equation for ϵ_y: $\quad \epsilon_y \cong -\dfrac{I_z \dot{\epsilon}_z}{\omega_0 (I_y - I_x)}$

$$\Rightarrow \dot{\epsilon}_y \cong -\frac{I_z \ddot{\epsilon}_z}{\omega_0 (I_y - I_x)}$$

$$\Rightarrow \ddot{\epsilon}_z = \epsilon_z \omega_0^2 \frac{(I_y - I_x)(I_x - I_z)}{I_y I_z}$$

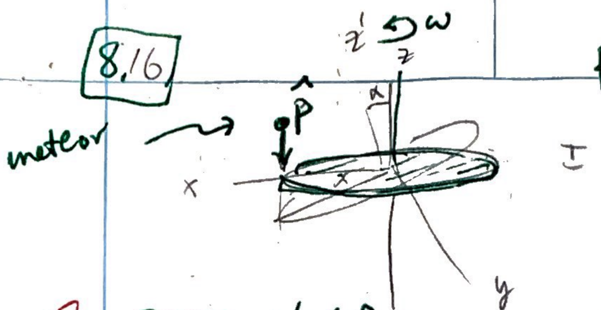
and, similarly, $\ddot{\epsilon}_y = \epsilon_y \omega_0^2 \dfrac{(I_y - I_x)(I_x - I_z)}{I_y I_z}$

So ϵ_y and ϵ_z are subject to a linear restoring force producing stability if $(I_y - I_x)(I_x - I_z) < 0$.

$\therefore$ Stable if I_x is greatest or smallest principle moment of inertia

Unstable if I_x is between the other two.

8.16



9

Find resulting motion after the meteor strikes platform

by conservation of angular mom

$$\hat{N}_y = rF = \frac{d\hat{L}}{dt}$$

during Δt:

$$N_y = I\dot{\omega}_y = aF = a\,\hat{P}/\Delta t$$

$$\boxed{\dot{\omega}_y \Delta t = \frac{a\hat{P}}{I}}$$

$$\int_0^{\Delta\omega_{y\,max}} d\omega_y = \int_0^{\Delta t} \frac{aF\,dt}{I} \qquad \underline{\underline{\hat{\omega}_y}} = \frac{aF\Delta t}{I} = \underline{\underline{\frac{a\hat{P}}{I}}}$$

$$\sin\alpha_{max} = \frac{\Delta z}{a} = \frac{\frac{1}{2}a\dot{\omega}_y\Delta t^2}{a} = \frac{1}{2}\dot{\omega}_y\Delta t^2$$

$$\alpha = \sin^{-1}\left(\frac{1}{2}\dot{\omega}_y\Delta t^2\right)$$

we assume spin remains the same: i.e. $\omega_z' = \omega_z = \omega$

z' precesses about z by $\Omega = \left(\frac{I_3}{I} - 1\right)\omega_z'\cos\alpha$

$$\underline{\underline{\Omega}} = (1)\,\omega\cos\left[\sin^{-1}\left(\frac{1}{2}\dot{\omega}_y\Delta t^2\right)\right] = \underline{\underline{\omega\sqrt{1-\frac{\dot{\omega}_y^2\Delta t^4}{4}}}} = \underline{\underline{\omega\sqrt{1-\frac{a^2\hat{P}^2}{I^2}\Delta t^2}}}$$

z precess about a fixed vertizle axis (invariable line)

by $\underline{\dot{\varphi}} = \omega\left(1+3\cos^2\alpha\right)^{1/2} = \omega\left(1+3\left(1-\frac{1}{4}\dot{\omega}_y^2\Delta t^4\right)\right)^{1/2}$

$$= \underline{\underline{\omega\left(1+3\left(1-\frac{1}{4}\frac{a^2\hat{P}^2}{I^2}\Delta t^2\right)\right)^{\frac{1}{2}}}}$$

please see the key

8.16) $\hat{P} = \hat{P}\hat{k}$, $\hat{L} = -\hat{P}a\hat{j}$ if the impulse is imparted on the x-axis at $a\hat{i}$.

$$\underline{\Delta\vec{v}_{cm} = \frac{\hat{P}}{m}\hat{k}}.$$

$$\Delta\vec{L} = \hat{L}$$

$$\vec{L}_{initial} = \underline{\underline{I}}\,\omega\hat{k} \qquad \underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\frac{ma^2}{4} \qquad \omega\hat{k} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

$$\Rightarrow \vec{L}_{initial} = \frac{m\omega a^2}{2}\hat{k}$$

$$\vec{L} = \vec{L}_{initial} + \hat{L} = \underline{\frac{m\omega a^2}{2}\hat{k} - \hat{P}a\hat{j}}$$ immediately after the impulse

Subsequent motion is torque free.

$$\Rightarrow I\dot{\omega}_x + \omega_y\omega_z(I_s - I) = 0$$

$$I\dot{\omega}_y + \omega_z\omega_x(I - I_s) = 0$$

$$I_s\dot{\omega}_z = 0$$

$$\vec{L} = \frac{ma^2}{4}\omega_x\hat{i} + \frac{ma^2}{4}\omega_y\hat{j} + \frac{ma^2}{2}\omega_z\hat{k}$$

$$\Rightarrow \omega_x = 0,\quad \omega_y = -\frac{4\hat{P}}{ma},\quad \omega_z = \omega.$$ immediately after the meteorite hits.

$$\left\{\begin{aligned}
&\omega_z = \text{const.} = \omega. \\
&\Omega = \omega \\
&\omega_x = +\frac{4\hat{P}}{ma}\sin\Omega t = +\frac{4\hat{P}}{ma}\sin\omega t \\
&\omega_y = -\frac{4\hat{P}}{ma}\cos\Omega t = -\frac{4\hat{P}}{ma}\cos\omega t \\
&\tan\alpha = \frac{4\hat{P}}{ma\omega}. \\
&\dot{\phi} = \sqrt{\omega^2 + \frac{16\hat{P}^2}{m^2a^2}}\sqrt{1 + 3\cos^2\alpha}
\end{aligned}\right.$$

diam = b

8.21

(a) $\omega = ?$ to keep the pencil upright ?

pencil spins in upright position

we let $u_3 = 1$ so $a = \omega_3 \theta$ $\theta = 0$

9

stability: $S_{pin}^2 > \dfrac{4 I m g \left(\frac{a}{2}\right)}{I_s^2}$

or $S^2 > \left(\dfrac{4\, mg \left(\frac{a}{2}\right)\left(\frac{b^2}{16} + \frac{a^2}{12}\right) m}{m^2 \frac{(b^2)^2}{64}} \right)$

$> 128\, a g \left(\frac{1}{16} + \frac{a^2}{b^2\, 12} \right)$ close

(b) for $a = 20$ $b = 1$

$S \approx 128 (20)(980)\left(\frac{20^2}{12}\right)$ r/sec

$\approx$

$S > 1.7 \times 10^5$ rpm

Roy W Erickson
Oct 19, 81
528-13-8795

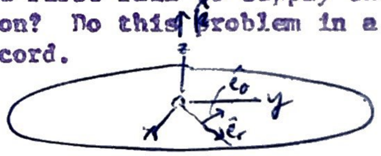
PHYSICS 321 TAKE-HOME QUIZ Oct. 16, 1981

Due: **1:10 p.m., Oct. 19**

Open book - you may use your text, notes, and problem solutions. You may also use math tables. You may not use any other books or consult with anyone except Dr. Evenson.

(10) 1. A bug walks with constant speed v outward from the center along a radius of a phonograph record which is turning with constant angular velocity ω. The coefficient of friction between bug and record is μ. At what radius will the friction force first fail to supply the needed centripetal acceleration? Do this problem in a coordinate system fixed to the record.

10/10



$$\boxed{\vec{r} = r\hat{e}_r}$$

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \qquad \dot{\theta} = 0$$

$$\boxed{\dot{\vec{r}} = \dot{r}\hat{e}_r} \qquad \boxed{\vec{\omega} = \omega\hat{k}}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta = \vec{0} \qquad \boxed{\vec{A}_0 = 0}$$

$$\boxed{\ddot{\vec{r}} = \vec{0}}$$

$$\ddot{\vec{R}} = \ddot{\vec{r}} + \vec{A}_0 + 2\vec{\omega}\times\dot{\vec{r}} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\vec{A} = \ddot{\vec{R}} = \vec{0} + \vec{0} + 2(\omega)(v)\underbrace{(\hat{k}\times\hat{e}_r)}_{\hat{e}_\theta} + \vec{0} + \omega^2 r\underbrace{(\hat{k}\times\underbrace{(\hat{k}\times\hat{e}_r)}_{\hat{e}_\theta})}_{-\hat{e}_r}$$

$$\ddot{\vec{R}} = 2\omega v\hat{e}_\theta - \omega^2 r\hat{e}_r$$

$$\vec{F}_f = m\ddot{\vec{R}} \equiv \mu m g\frac{\ddot{\vec{R}}}{|\ddot{\vec{R}}|} \qquad \text{or} \qquad |\ddot{\vec{R}}| = \mu g = \sqrt{(2\omega v)^2 + (-\omega^2 r)^2}$$

$$\text{so } + \frac{\sqrt{(\mu g)^2 - 4\omega^2 v^2}}{\omega^2} = r$$

$$\boxed{\sqrt{\frac{(\mu g)^2 - 4(\omega v)^2}{\omega^4}} = r}$$